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On the Use of the Method of Quadrature by Differentiation for Solving Eigenvalue Problems in Hydrodynamic Stability

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The usefulness of the method of quadrature by differentiation for solving eigenvalue problems in hydrodynamic stability is demonstrated. A numerical example, the Taylor problem, is presented in order to show the efficacy of the method and provide a basis of comparison with other approximate methods (e.g. the Galerkin method).

The results were found to be in good agreement with experimental data and it was demonstrated that the method of quadrature by differentiation in comparison with other analytical methods requires no trial and error, no extensive mathematical investigation, and no lengthy computer computations.

1. INTRODUCTION

The purpose of this paper is to demonstrate the usefulness of the infrequently used method of quadrature by differentiation [1] for solving eigenvalue problems in hydrodynamic stability. A numerical example is chosen in order to show the ease of application of the method. The method, however, is applicable to much more complicated problems.

It is useful to show the application of the method to an eigenvalue problem in hydrodynamic stability that has already been solved by other methods. In this way, a basis of comparison is established between quadrature by differentiation and other approximate methods (the Galerkin method in particular).

It will be demonstrated that the method consists of a number of straightforward

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steps demanding no trial functions, no trial and error, and no hindsight. On the other hand, the Galerkin method, a subset of the method of weighted residuals, requires a choice of trial functions and thus can require a considerable amount of physical insight and trial and error (for example, see [2]). However, the real advantage of the method lies in its applicability to nonlinear, as well as linear differential equations, and nonself-adjoint as well as self-adjoint differential equations.

2. Physical Problem

The problem to be examined is that of the stability of viscous, incompressible flow between concentric rotating cylinders, i.e., the Taylor problem. The cylinders are assumed to be infinitely long and rotate in opposite directions.[1] The inner and outer radii are given by R_1 and R_2, respectively. The corresponding angular velocities are given by Ω_1 and Ω_2. In this illustrative problem, the "small gap" case $[(R_2 - R_1) \ll (R_1 + R_2)/2]$ will be considered.

The equations of motion yield the stationary solution for the velocity in the transverse direction

$$V = Ar + B/r \qquad (R_1 \leqslant r \leqslant R_2), \tag{1}$$

where

$$A = \Omega_1(1 - \Omega_2 R_2^2/\Omega_1 R_1^2)/(1 - R_2^2/R_1^2)$$

and

$$B = \Omega_1 R_1^2(1 - \Omega_2/\Omega_1)/(1 - R_1^2/R_2^2).$$

Following the standard analysis, a disturbance is superimposed on the basic flow and the resultant pressure and velocities are substituted into the Navier–Stokes equations. After noting that the basic flow itself satisfies these equations, a set of nonlinear partial differential equations for the disturbance quantities is obtained. This set of equations is linearized in the disturbances by neglecting terms higher than the first order in the disturbances and their derivatives. The justification for the linear analysis lies in the fact that it has predicted results which are in excellent agreement with experimental evidence in numerous cases.

The linearized partial differential equations are solved using a normal mode analysis. This analysis uses disturbance functions with separated variables having

[1] The case in which the cylinders rotate in opposite directions generally presents more formidable mathematical and computational problems.

rotational symmetry and periodicity in the axial direction. The resulting ordinary differential equation is as follows.

$$(DD^* - \lambda^2)^3\, v = (4A\lambda^2/\nu^2)(A + B/r^2)\, v, \tag{2}$$

where $D = d/dr$, and $D^* = d/dr + 1/r$. Here $v = v(r)$ is the component of the disturbance velocity in the radial direction, ν is the kinematic viscosity, and $2\pi/\lambda$ is the wavelength of the disturbance in the axial direction. The boundary conditions are

$$\left.\begin{aligned} v &= 0, \\ (DD^* - \lambda^2)\, v &= 0, \\ D^*(DD^* - \lambda^2)\, v &= 0, \end{aligned}\right\} \text{ at } r = R_1 \text{ and } r = R_2 .$$

If the gap is small compared with the mean of the two radii, then the above equation reduces to

$$(D^2 - \lambda^2)^3\, v = (4A\lambda^2/\nu^2)(A + B/r^2)\, v, \tag{3}$$

with boundary conditions,

$$\left.\begin{aligned} v &= 0, \\ (D^2 - \lambda^2)\, v &= 0, \\ D(D^2 - \lambda^2)\, v &= 0, \end{aligned}\right\} \text{ at } r = R_1 \text{ and } r = R_2 .$$

With the change in variable, (following [2])

$$\begin{aligned} d_1 &= R_0 - R_1, \\ R_0 \text{ is defined as } r \text{ where } \quad V(r) &= 0, \\ r &= R_0 + \xi, \\ x &= (d_1 + \xi)/d_1, \\ a &= \lambda d_1, \\ S &= 8A^2 d_1^5/\nu^2 R_0 . \end{aligned}$$

Equation (3) becomes

$$(D^2 - a^2)^3\, v(x) = Sa^2(x - 1)\, v(x), \tag{4}$$

with boundary conditions,

$$\left.\begin{aligned} v &= 0, \\ (D^2 - a^2)\, v(x) &= 0, \\ D(D^2 - a^2)\, v(x) &= 0, \end{aligned}\right\} \text{ at } x = 0 \text{ and } x = d/d_1 ,$$

where $d = R_2 - R_1$ and $D = d/dx$ and the term V/r has been approximated by a straight line profile which has the slope of V/r at R_0.

This represents the complete formulation of the eigenvalue problem to be considered in this paper. The problem is to determine the values of "S" and "a" for which the differential equation has a solution. The values of "S" and "a" represent physically (i.e., are related to) the critical Reynolds (or Taylor) number at which the flow becomes unstable and the axial spacing of the disturbances, respectively.

This problem, as is well known, was first studied both experimentally and theoretically by Taylor [3]. Taylor's theoretical solution employed the expansion of the disturbance velocities in a series of orthonormal functions made up of first-order Bessel functions. His results were in excellent agreement with experimental evidence and this work represented the first successful application of linear hydrodynamic stability theory. At a later date, theoretical and experimental results further confirmed his conclusions.

Before Taylor's pioneering work on the stability of the flow between long concentric rotating cylinders, Couette [4], Mallock [5, 6] and Rayleigh [7] had investigated this circular flow. Couette and Mallock's work involved experimentally determining the drag on one cylinder while the other rotated. Rayleigh investigated the stability of inviscid Couette flow and established his now famous stability criterion, that the necessary and sufficient condition for stability with respect to rotationally symmetric disturbances is that $(d/dr)(r^2\Omega)^2 > 0$. Here $\Omega(r)$ is the angular velocity of rotation of the fluid at a radius r. Later Synge [8] considered the viscous case and has shown that Rayleigh's criteria is a sufficient condition for stability but not a necessary one for a viscous circular flow.

Since Taylor's work, many other researchers have theoretically investigated the Taylor problem, including Chandrasekhar [9, 10], Kirchgassner [11], Chandrasekhar and Elbert [12], Duty and Reid [13], Harris and Reid [14], Sparrow, Munro and Jonsson [15], Meksyn [16, 17, 18], Walowit, Tsao and DiPrima [19], DiPrima [2], Yu and Sun [20], Roberts [21], and Meyer [22], using a variety of methods including expansion procedures, integral methods, solution of adjoint systems of equations, asymptotic expansions, the Galerkin method, and numerical methods. However, none of these methods enjoy the ease of application of the method of quadrature by differentiation.

3. Analysis

The quadrature formulas ([1]) enable one to approximate the integral of a function, $f(x)$, over a given interval by simply utilizing the values of the function and its derivatives at each endpoint. For three-term quadrature, the expression would be

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}[f(0)+f(1)] + \frac{1}{10}[f'(0)-f'(1)] + \frac{1}{120}[f''(0)+f''(1)]. \quad (5)$$

For the problem under consideration, three-term quadrature proved to be adequate for values of "S" in the range of greatest change and was used throughout. In general, the number of terms retained in the quadrature formulas is determined by the desired degree of accuracy of the approximation.

In order to apply the method of quadrature by differentiation to the problem at hand, the differential equation was first normalized to the interval [0, 1] by utilizing the change of variable $x = d/d_1 y$. The differential equation becomes

$$\left(\frac{d_1}{d}\right)^6 v^{\mathrm{VI}}(y) - 3a^2\left(\frac{d_1}{d}\right)^4 v^{\mathrm{IV}}(y) + 3a^4\left(\frac{d_1}{d}\right)^2 v''(y)$$

$$+ \left[Sa^2\left(1 - \frac{d}{d_1}y\right) - a^6\right] v(y) = 0, \quad (6)$$

with boundary conditions,

$$\left.\begin{aligned} v &= 0, \\ v'' - a^2(d/d_1)^2 v &= 0, \\ v''' - a^2(d/d_1)^2 v' &= 0, \end{aligned}\right\} \text{ at } y = 0 \text{ and } y = 1.$$

At the endpoint $y = 0$, v is represented by the expansion

$$v = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \cdots. \quad (7)$$

At $y = 1$, v is represented by the expansion

$$v = C_0 + C_1(y-1) + C_2(y-1)^2 + C_3(y-1)^3 + \cdots. \quad (8)$$

Substitution of these expansions into the differential equation and application of the boundary conditions yields relationships among the coefficients b_i and relationships among the coefficients C_i. These relationships, obtained by satisfying the differential equation and the boundary conditions, are as follows.

$$b_0 = 0, \qquad C_0 = 0,$$

$$b_2 = 0, \qquad C_2 = 0,$$

$$b_3 = (1/6)\,a^2\beta^2 b_1, \qquad C_3 = (1/6)\,a^2\beta^2 C_1,$$

$$b_6 = (1/30)\,K_1 b_4, \qquad C_6 = (1/30)\,K_1 C_4,$$

$$b_7 = \frac{120K_1 b_5 - (K_2 a^2\beta^2 + K_3)\,b_1}{5040}, \qquad C_7 = \frac{120K_1 C_5 - (K_2 a^2\beta^2 + K_3 - K_4)\,C_1}{5040},$$

$$b_8 = \frac{12K_1^2 b_4 - 12K_2 b_4 + K_4 b_1}{20160}, \qquad C_8 = \frac{12K_1^2 C_4 - 12K_2 C_4 + K_4 C_1}{20160},$$

where $K_1 = 3a^2\beta^2$, $K_2 = 3a^4\beta^4$, $K_3 = Sa^2\beta^6 - a^6\beta^6$, $K_4 = Sa^2\beta^7$, and $\beta = d/d_1$. In addition, the three-term quadrature formulas for $v'(y)$, $v''(y)$, $v^{\mathrm{IV}}(y)$, and $v^{\mathrm{VI}}(y)$ were then invoked and equations relating the b_i's to the C_i's were obtained. These results (which utilize the fact that $b_0 = C_0 = b_2 = C_2 = 0$) are as follows:

$$\begin{aligned} b_1 + (b_3/10) &= -(C_1 + (C_3/10)) \\ C_1 - b_1 &= (3/5)(b_3 - C_3) + (1/5)(b_4 + C_4), \\ C_3 - b_3 &= 2(b_4 + C_4) + 2(b_5 - C_5) + (b_6 + C_6), \\ 12(C_5 - b_5) &= 360(b_6 + C_6) + 504(b_7 - C_7) + 336(b_8 + C_8). \end{aligned}$$

After some algebraic manipulation, the equations may be reduced to two equations for $(b_4 + C_4)/b_1$. By eliminating this expression, the following characteristic equation

$$1200 + 160a^2\left(\frac{d}{d_1}\right)^2 - \frac{1}{5}\left[Sa^2\left(\frac{d}{d_1}\right)^6 - a^6\left(\frac{d}{d_1}\right)^6\right] + \frac{1}{10}Sa^2\left(\frac{d}{d_1}\right)^7 + 6a^4\left(\frac{d}{d_1}\right)^4 = 0 \tag{9}$$

was obtained. Therefore,

$$S = \frac{\frac{1}{5}a^6\left(\frac{d}{d_1}\right)^6 + 6a^4\left(\frac{d}{d_1}\right)^4 + 160a^2\left(\frac{d}{d_1}\right)^2 + 1200}{\frac{1}{5}a^2\left(\frac{d}{d_1}\right)^6 - \frac{1}{10}a^2\left(\frac{d}{d_1}\right)^7}. \tag{10}$$

The values of "a" for which "S" is a minimum are desired. Setting $dS/da = 0$ and solving for "a" (with $d^2S/da^2 > 0$) yields

$$ad/d_1 = 3.284.$$

Thus for given values of the parameter d_1/d, the corresponding values of "S" and "a" are determined once and for all. Therefore,

$$S = 3592(d_1/d)^5/[2(d_1/d) - 1] \tag{11}$$

and $a = 3.284(d_1/d)$. These results along with the analytical results of DiPrima [2] and the experimental results of Taylor [3] are displayed in Fig. 1. The characteristic stability curves of "S" versus "a" are displayed in Fig. 2.

The method gives quite accurate values of "S" in the range of greatest change (and greatest interest), $0.6 \leqslant d_1/d \leqslant 1.0$, as shown in Fig. 1. For the range of $d_1/d < .6$, in which "S" is expected to be fairly constant, the results indicated that three-term quadrature was not adequate.

The physical grounds for expecting that "S" would be fairly constant in the region $0 < d_1/d < .6$ is closely tied to the location of R_0. Below values of $d_1/d = 0.5$, R_0 is closer to the inner cylinder. If R_0 is close to the inner cylinder, the position of the outer cylinder would have less effect on the instability because the centrifugal

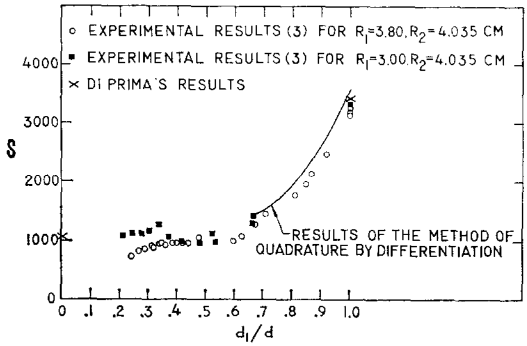

FIG. 1. Comparison between the results of the method of quadrature by differentiation and the experimental results of Taylor [3] and the theoretical predictions of DiPrima [2].

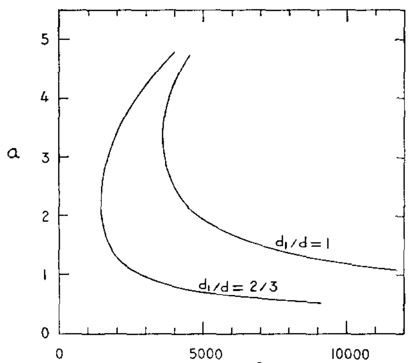

FIG. 2. Characteristic stability curves obtained by the method of quadrature by differentiation.

force tends to cause instability in the region $R_1 < r < R_0$ while tending to stabilize the flow in the region $R_0 < r < R_2$. With this in mind, DiPrima [2] restated the eigenvalue problem for $d_1/d \to 0$ by letting $R_2 \to \infty$. This requires that his eigenfunctions satisfy only the three boundary conditions at $x = 0$ and that they decay exponentially as $r \to \infty$. His solution for this case is also shown in Fig. 1.

4. Summary and Conclusion

As noted earlier, the case in which the cylinders rotate in opposite directions generally presents more formidable mathematical and computational problems. Taylor's original analytical solution still applied for this case, however, the calculations become much more tedious. DiPrima's solution (shown as X in Fig. 1), while it enjoys the relative simplicity of the Galerkin method, still requires an extensive search for suitable trial functions. Meksyn's asymptotic representations for this case are in good agreement with experimental results although his analysis requires a rather extensive mathematical investigation. The results of the method of quadrature by differentiation presented herein compare favorably with experimental results as shown in Fig. 1. The method, in comparison with the analytical results of DiPrima and Meksyn, requires no trial and error, no lengthy mathematical or computer computations, and no hindsight. On the other hand, the Galerkin method requires a considerable amount of trial and error in the selection of an adequate trial function (for different values of d_1/d, it would be expedient to use different trial functions) and more algebraic and computer computations. In addition, the method used in this paper is applicable to more complicated problems involving nonlinear and nonself-adjoint systems.

Difficulties in obtaining accurate experimental data and the possibility of an error in the data for $R_1 = 3.80$ were noted in [2, 17]. The question arose due to the good agreement of Meksyn's results in all cases but $R_1 = 3.80$. The problem in his analytical results occurs because of the sensitivity of the results to the accuracy of the measurement of R_1 when the distance between the cylinders is small and $-\omega$ ($\omega = \Omega_2/\Omega_1$) is large. The suggested possibility of an error in the experimental data is in the direction of higher values of "S" at given d_1/d which would increase the general agreement between the analytical solution presented herein and the experimental results.

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